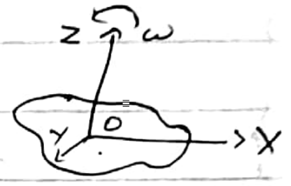
* Moment of Inertia –

Moment of Inertia, about a given axis plays the same part in rotational motion about that axis as the mass of a body does in translational motion.

" Mass is taken to be a measure of Inertia for linear or translatory motion."



$$I = \sum_i m_i r_i^2 = MK^2$$

$M \rightarrow$ total mass of the body

$K \rightarrow$ Effective distance of its particle from the axis of rotation.

If body does not consists discrete particles then,

$$I = \int r^2 dm = MK^2$$

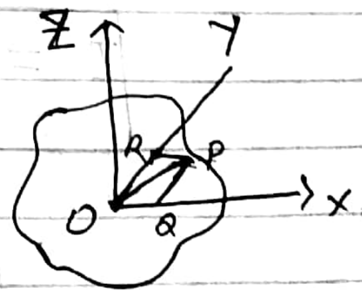
$dm \rightarrow$ mass of the infinitesimal small element of the body at a distance r from the axis.

General Theorems on Moment of Inertia –

(1) Theorem of $\perp^{ar}$ axis

Applicable to plane bodies.

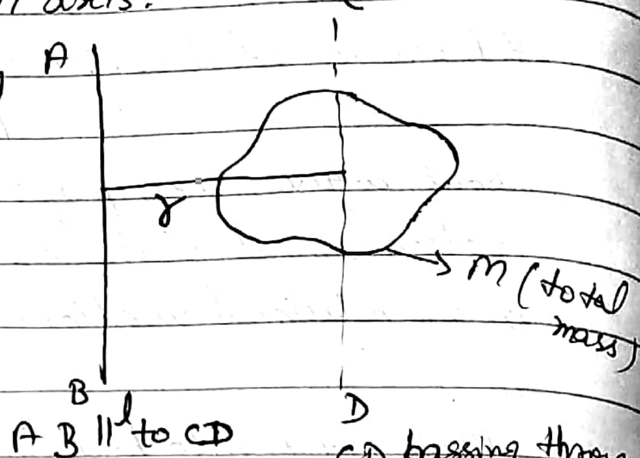
X and Y axis choosen ~~in plane~~ in plane of body and Z-axis $\perp^{ar}$ to this plane.



$$\boxed{I_Z = I_x + I_y}$$

(2) Theorem of $\parallel^l$ axis:—

$$\boxed{I = I_{cm} + Mr^2}$$

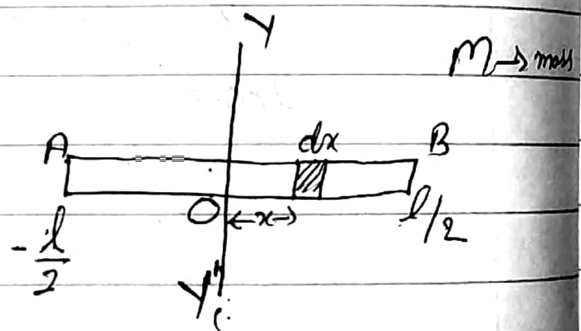


* Particular Cases of Moment of Inertia—

[1] M.I. of a Uniform rod of Length 'l'

(i) About an axis passing through its Centre and $\perp^{ar}$ to its length

$$I = \int_{-\frac{l}{2}}^{l/2} \frac{M}{l} dx \cdot x^2$$



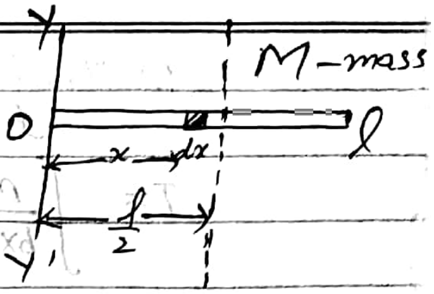
$$= 2\int_{0}^{l/2} \frac{M}{l} x^2 dx$$

$$\boxed{I = \frac{Ml^2}{12}}$$

(ii) About an axis passing through its one end and $\perp^{ar}$ to its length.

$$I = I_{cm} + Mx^2$$

$$I = \frac{Ml^2}{12} + M\left(\frac{l}{2}\right)^2$$



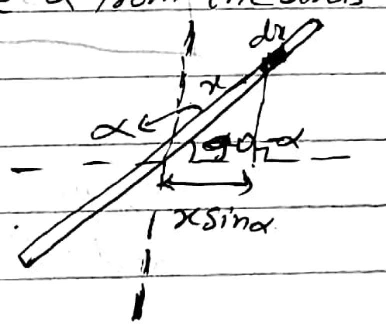
(Passing through centre of mass)

$$\boxed{I = \frac{Ml^2}{3}}$$

or

$$I = \int_0^l \frac{M}{l} \cdot dx \cdot x^2 = \frac{Ml^2}{3}$$

(iii) If rod is aligned at angle α from the axis of rotation -



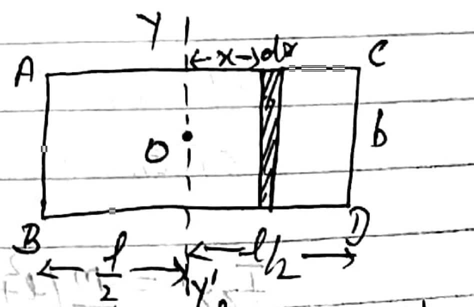
$$I = \int_{-l/2}^{l/2} \frac{M}{l} dx (x \sin\alpha)^2$$

$$\boxed{I = \frac{Ml^2}{12} \sin^2\alpha}$$

[2] Moment of Inertia of a rectangular lamina (bar)

(i) about an axis through its centre and $\parallel^l$ to its one side

$$I = \int_{-l/2}^{l/2} \left(\frac{M}{l \times b} \times b\,dx\right) \cdot x^2$$

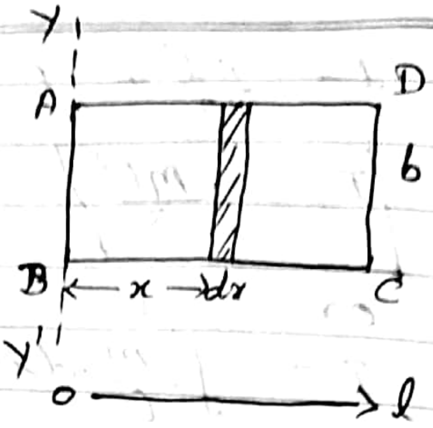


($\parallel^l$ to AB and CD)

$$\boxed{I = \frac{Ml^2}{12}}$$

If YOY' is $\parallel^l$ to AC & BD then $I = \frac{Mb^2}{12}$

(ii) About one side.



$$I = \int_0^l \frac{m}{b \times l} \cdot b\,dx \cdot x^2$$

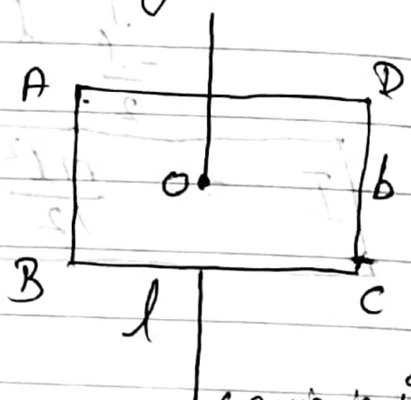
$$= \int_0^l \frac{m}{l} \cdot x^2\,dx$$

$$\boxed{I = \frac{ml^2}{3}}$$

or

$$\boxed{I = \frac{ml^2}{12} + m\left(\frac{l}{2}\right)^2}$$

(iii) About an axis passing through its centre and $\perp^{ar}$ its plane:–



Theorem of $\perp^{ar}$ axis

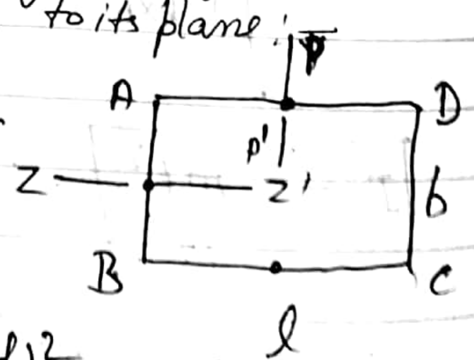
$$I_z = I_x + I_y$$

$$I = \frac{ml^2}{12} + \frac{mb^2}{12}$$

$$\boxed{I = \frac{1}{12} m(l^2 + b^2)}$$

(axis is $\perp^{ar}$ to ABCD plane)

(iv) About an axis passing through the mid point of one side and $\perp^{ar}$ to its plane:



$$I_{zz'} = \frac{m}{12}(l^2 + b^2) + m\left(\frac{b}{2}\right)^2$$

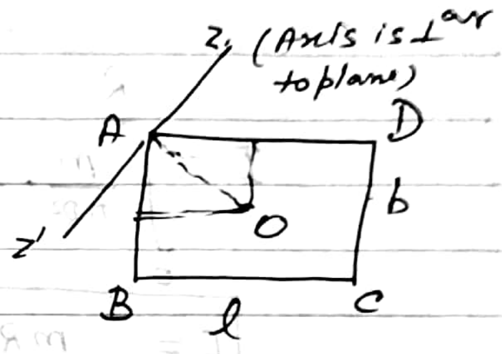
and

$$I_{pp'} = \frac{m}{12}(l^2 + b^2) + m\left(\frac{l}{2}\right)^2$$

passing through

→ Axis ZZ' is $\perp^{ar}$ to ABCD plane and passing through mid point of AB it may be CD also

→ Similarly PP' is $\perp^{ar}$ to plane and passing through mid point of AD may be BC also.



(V) $\because AO^2 = \left(\frac{l}{2}\right)^2 + \left(\frac{b}{2}\right)^2$

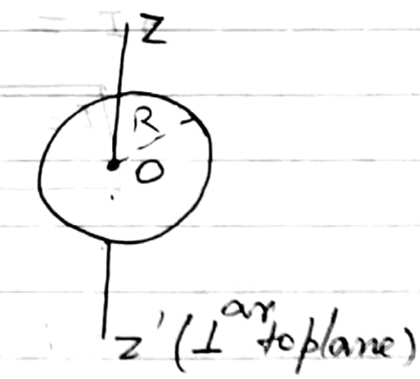
$$= \frac{l^2+b^2}{4}$$

$$I = I_{cm} + m(AO)^2$$

$$\boxed{I = \frac{m}{12}(l^2+b^2) + \frac{m}{4}(l^2+b^2)}$$

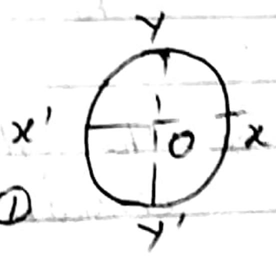
[3] Moment of Inertia of a hoop or a thin circular ring

(i) About an axis passing through its centre and $\perp^{ar}$ to its plane.



$$I = MR^2$$

(ii) About its diameter:-



By theorem

$$I_x + I_y = I_z \quad \text{—} ①$$

$$\because I_x = I_y$$

$$\boxed{I_x = I_y = \frac{I_z}{2} = \frac{MR^2}{2}}$$

[4] Moment of Inertia of a circular lamina or disc

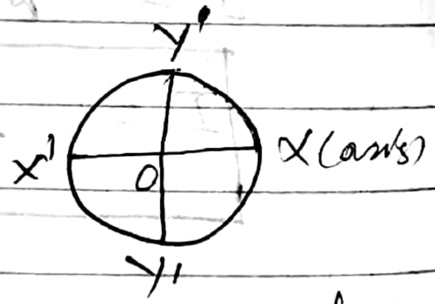
(i) About an axis through centre and $\perp^{ar}$ to plane –

(axis is through centre O & $\perp^{ar}$ to plane

$$I = \int_0^R \frac{M}{\pi R^2} \cdot 2\pi x \, dx \cdot x^2$$

$$\boxed{I = \frac{MR^2}{2}}$$

(ii) About a diameter: –



$I_x = I_y = I$ (M.I. about any diameter is same)

$I_x + I_y = I_z$

I_z is M.I. about an axis $\perp^{ar}$ to plane & passing through O

$2I = \frac{1}{2} MR^2$

$$\boxed{I = \frac{MR^2}{4}}$$